L25 An Example

1. The problems and initial analysis

   (1) The problems
   There are two Poisson populations with means λ_1 and λ_2. We want to compare the two means. Specifically we want to test

   $$\boxed{H_0: \lambda_1 \le \lambda_2 \text{ versus } H_a: \lambda_1 > \lambda_2} \quad \boxed{H_0: \lambda_1 \ge \lambda_2 \text{ versus } H_a: \lambda_1 < \lambda_2}$$
   $$\boxed{H_0: \lambda_1 = \lambda_2 \text{ versus } H_a: \lambda_1 \ne \lambda_2}$$

   (2) Initial analysis
   Let $X_1, ..., X_{n_1}$ be a random sample from Poisson(λ_1) and $Y_1, ..., Y_{n_2}$ be a random sample from Poisson(λ_2). Then the joint pmf of two samples is

   $$\begin{aligned}
   &\prod_{i=1}^{n_1}\left(\frac{\lambda_1^{x_i}}{x_i!}e^{-\lambda_1}\right)\prod_{j=1}^{n_2}\left(\frac{\lambda_2^{y_j}}{y_j!}e^{-\lambda_2}\right) = \frac{1}{\prod_i x_i! \prod_j y_j!}\,e^{-n_1\lambda_1 - n_2\lambda_2}\,\lambda_1^{\sum_i x_i}\lambda_2^{\sum_j y_j}\\
   = \;& \exp\left[p(\lambda_1, \lambda_2) + q(x_1, ..., x_{n_1}, y_1, .., y_{n_2}) + (\ln\lambda_1)(\textstyle\sum_{i=1}^{n_1} x_i) + (\ln\lambda_2)(\sum_{j=1}^{n_2} y_j)\right]\\
   = \;& \exp\left[p(\lambda_1, \lambda_2) + q(x, y) + (\ln\lambda_1)(\textstyle\sum_{i=1}^{n_1} x_i) - (\ln\lambda_2)(\sum_{i=1}^{n_1} x_i)\right.\\
   &\qquad\qquad\left. + (\ln\lambda_2)(\textstyle\sum_{i=1}^{n_1} x_i) + (\ln\lambda_2)(\sum_{j=1}^{n_2} y_j)\right]\\
   = \;& \exp\left[p(\lambda_1\,\lambda_2) + q(x, y) + \left(\ln\frac{\lambda_1}{\lambda_2}\right)(\textstyle\sum_{i=1}^{n_1} x_i) + (\ln\lambda_2)\left(\sum_{i=1}^{n_1} x_i + \sum_{j=1}^{n_2} y_j\right)\right].
   \end{aligned}$$

   (3) Reparameterization
   Let $\theta = \ln\frac{\lambda_1}{\lambda_2}$ and $\tau = \ln\lambda_2$. Then the joint pmf of samples becomes

   $$f(x, y; \theta, \tau) = \exp\left[p_*(\theta, \tau) + q(x, y) + \theta T(x) + \tau S(x, y)\right].$$

   where $T(X) = \sum_{i=1}^{n_1} X_i$ and $S(X, Y) = \sum_{i=1}^{n_1} X_i + \sum_{j=1}^{n_2} Y_j$. Also

   $$\boxed{H_0: \lambda_1 \le \lambda_2 \text{ versus } H_a: \lambda_1 > \lambda_2} \Longleftrightarrow \boxed{H_0: \theta \le 0 \text{ versus } H_a: \theta > 0}$$
   $$\boxed{H_0: \lambda_1 \ge \lambda_2 \text{ versus } H_a: \lambda_1 < \lambda_2} \Longleftrightarrow \boxed{H_0: \theta \ge 0 \text{ versus } H_a: \theta < 0}$$
   $$\boxed{H_0: \lambda_1 = \lambda_2 \text{ versus } H_a: \lambda_1 \ne \lambda_2} \Longleftrightarrow \boxed{H_0: \theta = 0 \text{ versus } H_a: \theta \ne 0}$$

2. Frameworks

   (1) One-sided tests with nuisance parameter
   Note that $\theta_1 < \theta_2 \Longrightarrow \frac{f(x, y; \theta_2, \tau)}{f(x, y; \theta_1, \tau)} = \frac{\exp[p(\theta_2, \tau)]}{\exp[p(\theta_1, \tau)]}\exp[(\theta_2 - \theta_1)T(x)]$
   is an increasing function of $T(x)$ for all $S(x, y)$.
   $S(X, Y)$ is sufficient and complete for τ. Thus we obtain conditional α-level UMP tests

$H_0: \theta \le 0$ versus $H_a: \theta > 0$	$H_0: \theta \ge 0$ versus $H_a: \theta < 0$
$\phi(T) = \begin{cases} 1 & T > c_1(S) \\ r & T = c_1(S) \\ 0 & T < c_1(S) \end{cases}$	$\phi(T) = \begin{cases} 1 & T < c_2(S) \\ r & T = c_2(S) \\ 0 & T > c_2(S) \end{cases}$
with $E_{\theta=0}[\phi(T)\vert S] = \alpha$.	with $E_{\theta=0}[\phi(T)\vert S] = \alpha$.

   (2) Two-sided test with nuisance parameter
   The following is conditional α-level UMP

$H_0: \theta = 0$ versus $H_a: \theta \neq 0$

$\phi(T) = \begin{cases} 1 & T < c_1(S) \text{ or } T > c_2(S) \\ r_i & T = c_i(S),\ i = 1,\ 2 \\ 0 & c_1(S) < T < c_2(S) \end{cases}$

with $E_{\theta=0}[\phi(T)|S] = \alpha$ and $E_{\theta=0}[T\phi(T)|S] = \alpha E_{\theta=0}(T|S)$.

3. Implementation

(1) Distributions of T, S and $(T,\ S)$

$T = \sum_{i=1}^{n_1} X_i \sim \text{Poisson}(n_1\lambda_1)$ with pmf $f_T(t;\ \lambda_1) = \frac{(n_1\lambda_1)^t}{t!} e^{-n_1\lambda_1}$.

$H = \sum_{j=1}^{n_2} Y_j \sim \text{Poisson}(n_2\lambda_2)$ with pmf $f_H(h;\ \lambda_2) = \frac{(n_2\lambda_2)^h}{h!} e^{-n_2\lambda_2}$.

$S = T + H \sim \text{Poisson}(n_1\lambda_1 + n_2\lambda_2)$ with pmf $f_S(s;\ \lambda_1,\ \lambda_2) = \frac{(n_1\lambda_1+n_2\lambda_2)^s}{s!} e^{-(n_1\lambda_1+n_2\lambda_2)}$.

T and $H = S - T$ are independent. So the joint pmf of $(T,\ S)$ is

$$\begin{aligned} f_{(T,\,S)}(t,\ s;\ \lambda_1,\ \lambda_2) &= P(T=t,\ S=s;\ \lambda_1,\ \lambda_2) = P(T=t,\ T+H=s;\ \lambda_1,\ \lambda_2) \\ &= P(T=t,\ H=s-t;\ \lambda_1,\ \lambda_2) = P(T=t,\ \lambda_1)\, P(H=s-t;\ \lambda_2) \\ &= P(\text{Possion}(n_1\lambda_1)=t)\, P(\text{Poisson}(n_2\lambda_2)=s-t) \\ &= \frac{(n_1\lambda_1)^t}{t!} e^{-n_1\lambda_1} \cdot \frac{(n_2\lambda_2)^{s-t}}{(s-t)!} e^{-n_2\lambda_2} \\ &= \frac{1}{t!\,(s-t)!} (n_1\lambda_1)^t\, (n_2\lambda_2)^{s-t} e^{-(n_1\lambda_1+n_2\lambda_2)}. \end{aligned}$$

(2) Conditional distribution of $T|S$ when $\theta = 0$

The conditional pmf of $T|S$, $f_{T|S}(t) = \frac{f_{(T,\,S)}(t,\,s)}{f_S(s)}$, is

$$\frac{s!}{t!\,(s-t)!} \left(\frac{n_1\lambda_1}{n_1\lambda_1+n_2\lambda_2}\right)^t \left(\frac{n_2\lambda_2}{n_1\lambda_1+n_2\lambda_2}\right)^{s-t} = \binom{s}{t} \left(\frac{n_1\lambda_1}{n_1\lambda_1+n_2\lambda_2}\right)^t \left(1 - \frac{n_1\lambda_1}{n_1\lambda_1+n_2\lambda_2}\right)^{s-t}.$$

So $T|S \sim \text{Binomial}\left(S,\ \frac{n_1\lambda_1}{n_1\lambda_1+n_2\lambda_2}\right)$.

Note that $\theta = 0 \Longleftrightarrow \lambda_1 = \lambda_2 \Longrightarrow \frac{n_1\lambda_1}{n_1\lambda_1+n_2\lambda_2} = \frac{n_1}{n_1+n_2}$ and $T|S \sim \text{Binomial}\left(S,\ \frac{n_1}{n_1+n_2}\right)$.

(3) An example

For $H_0: \lambda_1 \le \lambda_2$ versus $H_a: \lambda_1 > \lambda_2$ with level 0.05 we observed $n_1 = 10$, $n_2 = 15$, $T_{ob} = 5$ and $S_{ob} = 7$. So in

$$\phi(T) = \begin{cases} 1 & T > c \\ r & T = c \\ 0 & T < c \end{cases} \quad \text{with } 0.05 = E_{\theta=0}[\phi(T)|S]$$

$T|S \overset{\theta=0}{\sim} B\left(S_{ob},\ \frac{n_1}{n_1+n_2}\right) = B(7,\ 0.4)$ and

$$0.05 = \alpha = E_{\theta=0}[\phi(T)|S=7] = P(B(7,\ 0.4) > c) + rP(B(7,\ 0.4 = c)).$$

From $P(B(7,\ 0.4) > 5) = 0.01884 < 0.05$ and $P(B(7,\ 0.4) > 4) = 0.09626 > 0.05$, $c = 5$ and $r = \frac{0.05 - P(B(7,\,0.4)>5)}{P(B(7,\,0.4)=5)} = 0.4025$.

With $T_{ob} = 5$, we reject H_0 with probability 0.4025.

L26: Tests and confidence regions

1. From tests to confidence regions

(1) α-level test

For α-level test

$H_0 : \theta = \theta_0$ vs $H_a : \theta \neq \theta_0$ Test statistic: $T = T(X, \theta_0)$ Reject H_0 if $T \in A^c$ for significance level α

With significance level α, $P_{\theta_0}(T \in A^c) \leq \alpha \Longleftrightarrow P_{\theta_0}(T \in A) \geq 1 - \alpha$.

(2) $1 - \alpha$ confidence region for θ
$C(X)$ is a $1 - \alpha$ confidence region for $\theta \in R^k$
if $C(X)$ is a random region in R^k and $P_\theta(\theta \in C(X)) \geq 1 - \alpha$.

(3) Converting acceptance region to confidence region
From α-level test with test statistic $T(X, \theta_0)$ and acceptance region A, define $C(X)$ by

$$T(X, \theta_0) \in A \Longleftrightarrow \theta_0 \in C(X).$$

Then $C(X)$ is a $1 - \alpha$ confidence region for θ.
Proof. $P_\theta(\theta \in C(X)) = P_{\theta_0}(\theta_0 \in C(X)) = P_{\theta_0}(T(X, \theta_0) \in A) \geq 1 - \alpha$. So

$$T(X, \theta) \in A \text{ with significence level } \alpha \Longleftrightarrow \theta \in C(X) \text{ with confidence coefficient } 1-\alpha.$$

Ex1: From α-level t-test on μ in $N(\mu, \sigma^2)$

$H_0 : \mu = \mu_0$ vs $H_a : \mu \neq \mu_0$ Test statistic: $t = \frac{\overline{X}-\mu_0}{s/\sqrt{n}}$ Reject H_0 if $t < -t_{\alpha/2}(n-1)$ or $t > t_{\alpha/2}(n-1)$

find a $1 - \alpha$ confidence interval for μ.

$$\begin{aligned} t \in A \quad &\Longleftrightarrow \quad \frac{\overline{X}-\mu_0}{s/\sqrt{n}} \in \left[-t_{\alpha/2}(n-1),\, t_{\alpha/2}(n-1)\right] \\ &\Longleftrightarrow \quad \mu_0 \in \left[\overline{X} - t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}},\, \overline{X} + t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}}\right]. \end{aligned}$$

Thus $\overline{X} \pm t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}}$ is a $1 - \alpha$ confidence interval for μ.

2. Applications

(1) Is θ_0 in $1 - \alpha$ confidence region for θ?
Suppose $1 - \alpha$ confidence region $C(X)$ is generated from α-level test with test statistic $T(X, \theta_0)$. Based on

$$T(X, \theta) \in A \text{ with significence level } \alpha \Longleftrightarrow \theta \in C(X) \text{ with confidence coefficient } 1 - \alpha.$$

we need to test $H_0 : \theta = \theta_0$ versus $H_a : \theta \neq \theta_0$ at significance level α.
If H_0 is rejected, i.e., $T(X, \theta_0) \in A^c$, then θ_0 is not in $1 - \alpha$ C.R. for θ
If H_0 is accepted, i.e., $T(X, \theta_0) \in A$, then θ_0 is in the $1 - \alpha$ C. R. for θ.

(2) Find smallest confidence coefficient such that θ_0 is in the confidence region.

$$\begin{aligned} & \theta \in C(X) \text{ with confidence coefficient } 1-\alpha \\ \iff & T(X,\,\theta) \in A \text{ with significence level } \alpha \iff p\text{-value} > \alpha \\ \iff & 1 - p\text{-value} < 1-\alpha. \end{aligned}$$

Thus we need to test $H_0 : \theta = \theta_0$ vs $H_a : \theta \neq \theta_0$. Then the smallest confidence coefficient with which θ_0 is in the confidence region is $1 - (p\text{-value})$.

Comment: While computer can give p-value for a test quickly, the computation for confidence region might be tedious. For example with $\mu \in R^p$ in $N(\mu,\,\Sigma)$, $1-\alpha$ confidence region is given by

$$(\mu - \overline{X})' \left(\frac{S}{n}\right)^{-1} (\mu - \overline{X}) \le T^2_\alpha(p,\,n-1)$$

where $T^2_\alpha(p,\,n-1) = \frac{(n-1)p}{n-p} F_\alpha(p,\,n-p)$. So to get questions on confidence region answered via testing has practical significance.

3. Extension

With α-level test using test statistic $T = T(X,\,\theta_0)$ where $\theta_0 \in H_0$

$H_0 : \theta \in H_0$ vs $H_a : \theta \notin H_0$
Test statistic: $T = T(X,\,\theta_0)$
Reject H_0 if $T \in A^c$ for significance level α

Define $C(X)$ by $T(X,\,\theta_0) \in A \iff \theta_0 \in C(X)$. Then

$$\theta \in C(X) \text{ with cc} 1-\alpha \iff T(X,\,\theta_0) \in A^c \text{ with SL } \alpha$$

Proof. $P_\theta(\theta \in C(X)) = P_{\theta_0}(T(X,\,\theta_0) \in A) = 1 - P_{\theta_0}(T(X,\,\theta_0) \in A^c) \ge 1-\alpha$.

Ex2: From α-level lower-sided H_a t-test

$H_0 : \mu \ge \mu_0$ vs $H_a : \mu < \mu_0$
Test statistic: $T = \frac{\overline{X}-\mu_0}{s/\sqrt{n}}$
Reject H_0 if $T < -t_\alpha(n-1)$ for significance level α

$$T \in A \iff \frac{\overline{X}-\mu_0}{s/\sqrt{n}} \in (-t_\alpha(n-1),\,\infty) \iff \mu_0 \in \left(-\infty,\,\overline{X} + t_\alpha(n-1)\frac{s}{\sqrt{n}}\right).$$

Thus $\left(-\infty,\,\overline{X} + t_\alpha(n-1)\frac{s}{\sqrt{n}}\right)$ is a $1-\alpha$ lower-sided confidence interval for μ.

Ex3: Suppose

$H_0 : \mu \ge \mu_0$ vs $H_a : \mu < \mu_0$
Test statistic: $T = \frac{\overline{X}-\mu_0}{s/\sqrt{n}}$
p-value: $P(t(n-1) < t_{ob})$
p-value: 0.03.
Reject H_0 at the level 0.05.

Then μ_0 is not in the 95% lower-sided CI for μ

The smallest cc of CI containing μ_0 is $1 - 0.03 = 97\%$.